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ORIGINAL PAPER

Truncating and chamfering diagrams of regular polyhedra

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Abstract Truncating the vertex and chamfering the edges of a polyhedron is the classic way to obtain Archimedean polyhedra taking a Platonic solid as the starting point. We have considered the set of polyhedra obtained by this method and have found singular diagrams that show a special relation between these Archimedean polyhedra.

Keywords Archimedean polyhedra · Truncating · Chamfering · Octahedron · Icosahedron

1 Introduction

Platonic and Archimedean solids have attracted the attention of mathematicians and scientists of all times and places. Modern research [1] in the field of small metallic systems has now confirmed that many nanoparticles take Platonic and Archimedean solid-related shapes.

These polyhedra may be ordered by various numerical indicators according to their complexity: using the number of their vertices, the solid angle [2], via their Schlegel graphs [3], etc.

Truncating the vertex and chamfering the edges of a polyhedron is the classic way to obtain Archimedean polyhedra taking a Platonic solid as the starting point [4]. We have considered the set of polyhedra obtained from the regular octahedron and the

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regular icosahedron by means of this procedure, using truncating and chamfering planes perpendicular to their rotation axis, and have found singular diagrams that represent all these polyhedra.

These diagrams show a new visualization and classification of the set of polyhedra with parallel faces [5] which can be obtained from the regular polyhedra while maintaining all their symmetry elements (rotation axes and symmetry planes). Furthermore, this representation shows the method to construct these polyhedra by truncating and chamfering a Platonic solid.

2 Polyhedra derived from the regular octahedron

In a first step, we shall analyze the polyhedra derived from the regular octahedron by the sole truncation of its vertex. Then, in a second step, we shall consider two simultaneous operations: truncating the vertex and chamfering the edges.

In all cases, we shall only consider truncating and chamfering planes perpendicular to the rotation axes of the octahedron. The underlying rationale is to maintain all the symmetry elements of this regular polyhedron in order to obtain polyhedra with maximum symmetry.

2.1 Truncating the vertex of the regular octahedron

To identify the position of the truncating planes, we shall assign the distance (x) between the vertex to truncate and the truncating plane as shown in Fig. 1. To simplify, we shall suppose that the length of the edge of the octahedron is $a = 1$. Thus, the truncation value (x) represents per unit distances.

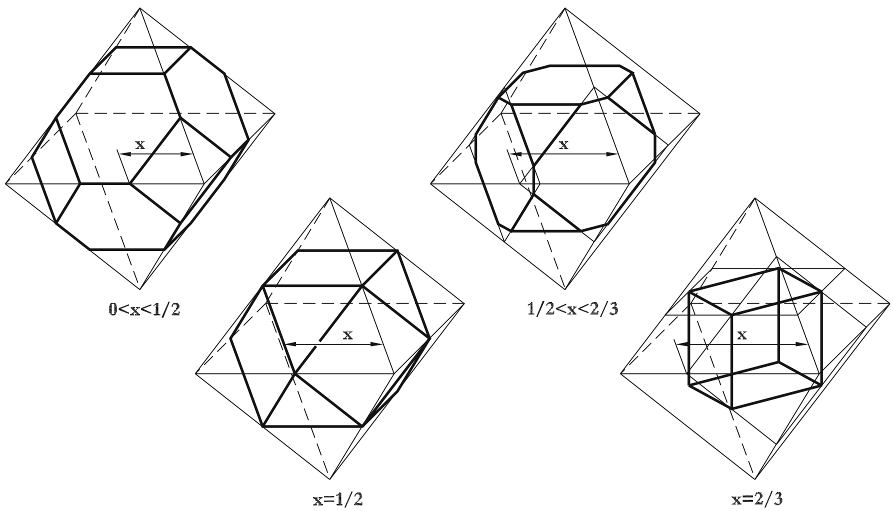

Fig. 1 Possible cases obtained by truncating the vertex of a regular octahedron

The resulting polyhedra when x goes from 0 to its maximum value ($x = 2/3$) are shown in Fig. 1.

These polyhedra can be classified in the following way:

For $0 < x < 1/2$, we obtain a polyhedron which has squares in place of the vertex of the octahedron and equiangular hexagons in place of its faces.

When $x = 1/2$, a triangle arises on each face of the initial octahedron and we obtain a cuboctahedron.

For $1/2 < x < 2/3$, we obtain a polyhedron with equiangular octagons in place of the vertex of the octahedron and triangles in place of the faces.

When $x = 2/3$, the truncation planes intersect one another at a point at the center of the face of the octahedron and we obtain a cube, which is dual to the octahedron. We consider this to be the maximum value of x; if we increase this value, the resulting polyhedron has no contact with the initial octahedron.

2.2 Truncating the vertex and chamfering the edges of the regular octahedron

The position of chamfering planes can be specified by the distance from the vertex (y) as shown in Fig. 2. We shall assume that the length of the edge of the octahedron is equal to one. Consequently, the value y represents per unit distances, as does the value x.

In order to show all possible cases, we established an arbitrary chamfering plane (given by the variable y) and then analyzed the resulting polyhedra when the truncation plane (represented by x) goes from the minimum ($x = 0$) to maximum value ($x = 2/3$) established previously. The process represented in Fig. 2 is the following:

For $0 < x < y$, the truncation plane does not cut the chamfered octahedron. That is to say, the truncating plane (given by x) is not relevant to the final result. The resulting polyhedron only depends on the variable y at this interval.

When $x = y$, the truncation plane touches the chamfered octahedron at a point.

For $y < x < 2y$, the truncating plane produce squares in place of the vertex of the octahedron and octagons in place of its edges.

When $x = 2y$, there are rectangles in place of the edges of the octahedron.

For $2y < x < (y + 1)/2$, there are equiangular octagons in place of the vertex and rectangles in place of the edges.

When $x = (y + 1)/2$, the rectangular faces situated in place of the edges of the octahedron disappear.

For $(y + 1)/2 < x < 2/3$, the resulting polyhedron has octagonal and triangular faces. In this interval, the chamfering plane does not touch the truncated octahedron. Thus, the effect of chamfering the edges is not relevant in this interval and the polyhedron so obtained only depends on the variable x.

When $x = 2/3$, we obtain a cube.

The maximum value for the chamfering (y) is 1/2. At that moment, the chamfering planes intersect one another at a point above the face of the octahedron, as shown in Fig. 2, and we obtain a rhombic dodecahedron.

All the cases shown can be represented in a diagram. The values (x, y) range within the following intervals

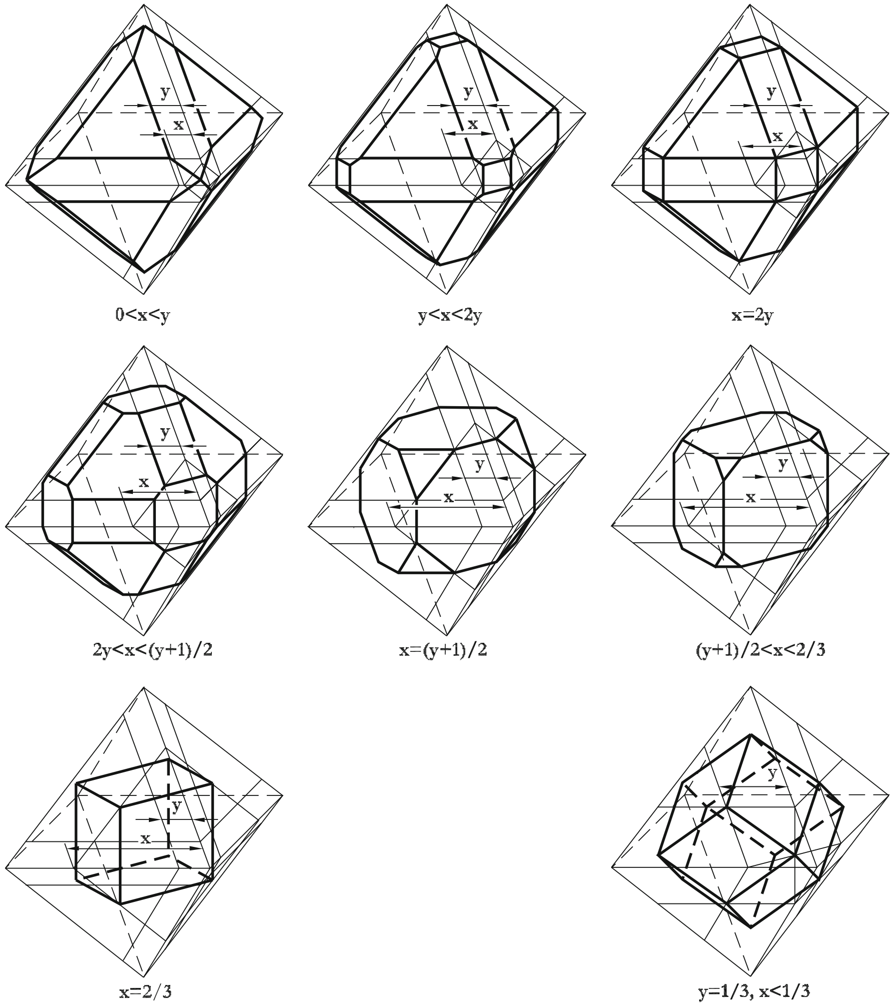

Fig. 2 Possible cases obtained by truncating the vertex and chamfering the edges of the regular octahedron

$$0 \le x \le 2/3, \quad 0 \le y \le 1/2$$

where the limits are represented by the cube (2/3, 0) and the rhombic dodecahedron (0, 1/2). The graphic representation of this interval is the rectangular region shown in Fig. 3. Each point inside this rectangle is associated with a pair of values (x, y) and therefore represents a particular polyhedron obtained from the regular octahedron by truncating its vertices and chamfering its edges.

According to the previously established classification, this rectangular region can be divided in four sub-regions, as shown in Fig. 3. Each of these sub-regions is associated with a type of polyhedron, as can be seen in the same figure.

The characteristics of the regions numbered 1 and 4 are worth highlighting.

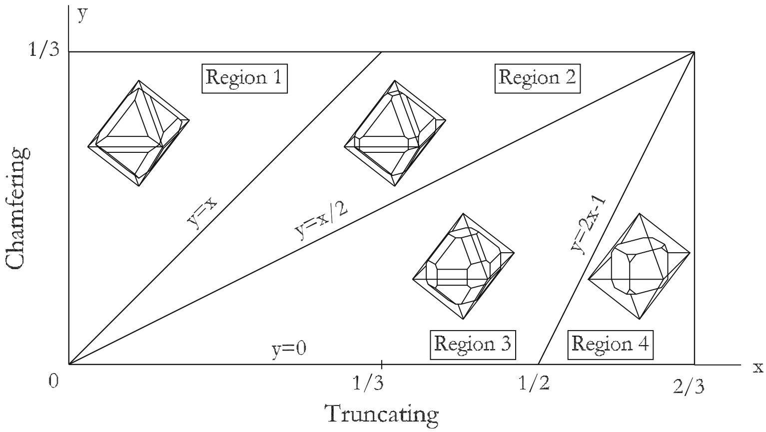

Fig. 3 Truncating and chamfering diagram of the regular octahedron

Region 1 is associated with polyhedral shapes obtained by only chamfering the edges of the octahedron; within this interval the truncation plane has no effect over the chamfered octahedron. Accordingly, each polyhedron is not associated with a single point in this region, but with a succession of points, which are represented by a horizontal segment on the aforementioned region.

In Region 4, however, the obtained polyhedra only depend on the truncation of the vertex and are independent of the chamfering planes. This means, in short, that no polyhedron is associated with a point, but rather with a vertical segment in this region.

2.3 Location of Archimedean polyhedra

We have calculated the suitable values of the pair (x, y) to obtain the seven Platonic and Archimedean polyhedra with octahedral symmetry. These values are shown in Table 1, while their representation over the truncating and chamfering diagram is plotted in Fig. 4.

We have noted that all these polyhedra are represented by points on a triangular area that coincides exactly with the region denominated as Region 3. At first sight,

Table 1 Truncating and chamfering values to obtain Archimedean polyhedra with octahedral symmetry

Polyhedron	Truncating (x)	Chamfering (y)
Octahedron	0	0
Truncated octahedron	1/3	0
Cuboctahedron	1/2	0
Truncated cuboctahedron	$\sqrt{2}/3$	$(\sqrt{2}-1)/3$
Rhombicuboctahedron	$2(3-\sqrt{2})/7$	$(3-\sqrt{2})/7$
Truncated cube	$2-\sqrt{2}$	0
Cube	2/3	0

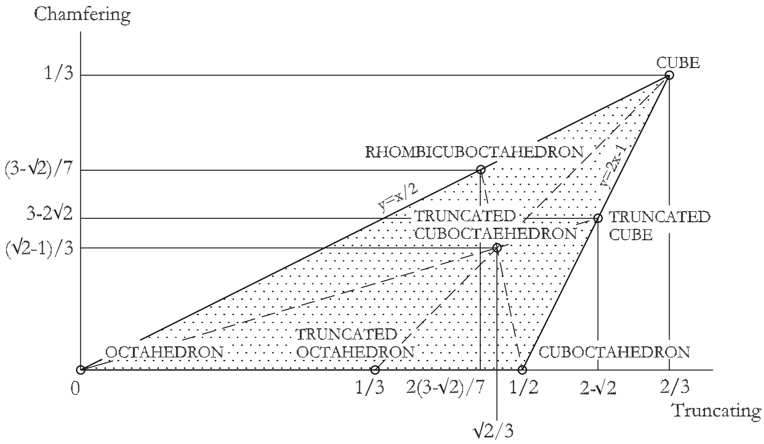

Fig. 4 Situation of the Archimedean polyhedra over the truncating and chamfering diagram

only the cube and the truncated cube would lie outside this region. However, bearing in mind that each polyhedron in Region 4 is represented by a vertical segment, it is acceptable to locate these two polyhedra on the boundary between Regions 3 and 4 (given by the straight line $y = 2x - 1$).

Accordingly all the Archimedean polyhedra can be represented inside Region 3, as shown in Fig. 4. This region is thus of great importance compared with the other three regions.

Detailed analysis reveals that all the points (x, y) which represent the Archimedean polyhedra in Region 3 are connected by straight lines that simultaneously pass through three points, as can be seen in Fig. 4.

The underlying reason for these connections is revealed by noting that all the faces of Archimedean polyhedra are regular polygons and every pair of parameters (x, y) that satisfies this regularity condition must lie on the set of six straight lines connecting the family of Archimedean polyhedra.

A singular case is the truncated cuboctahedron, which is located exactly at the intersection point of three straight lines. The explanation is that this polyhedron has three types of regular faces: squares, hexagons and octagons.

Similar results would be obtained if we begun with the regular hexahedron owing to the duality relation between this polyhedral shape and the regular octahedron.

3 Polyhedra derived from the regular icosahedron

In order to classify the polyhedra derived from the regular icosahedron by truncating the vertex and chamfering the edges, we performed a similar analysis to the one previously expounded for the regular octahedron using similar criteria.

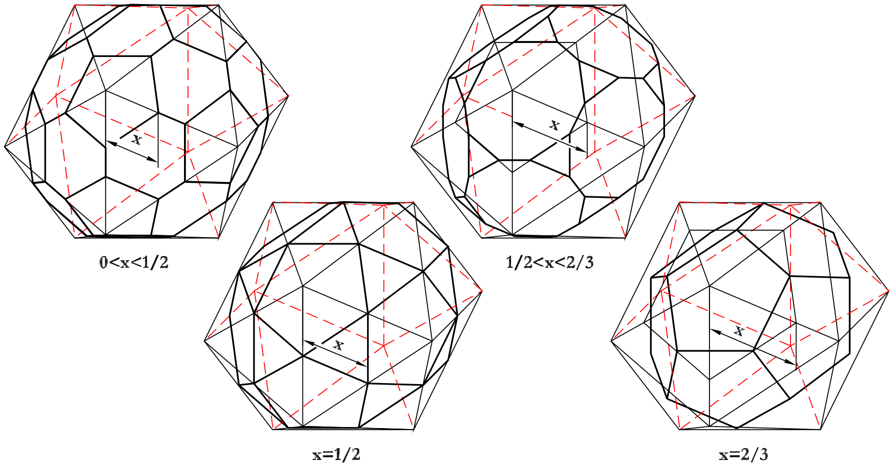

Fig. 5 Possible cases obtained by truncating the vertex of the regular icosahedron

3.1 Truncating the vertex of the regular icosahedron

To locate the truncating planes, we used the same criterion as that previously put forward for the octahedron. The possible cases are shown in Fig. 5.

3.2 Truncating the vertex and chamfering the edges of the regular icosahedron

All possible cases are shown in Fig. 6, while the resulting diagram, including its division in different regions, is plotted in Fig. 7. The obtained result is surprisingly similar to that obtained for the regular octahedron. In fact, the only difference between the two diagrams is the boundary limiting Regions 1 and 2.

3.3 Location of Archimedean polyhedra

The suitable values of x and y to obtain the Archimedean polyhedra with icosahedral symmetry, as well as the icosahedron and the dodecahedron, are shown in Table 2.

The location of these polyhedra over the truncating and chamfering diagram is shown in Fig. 8 with the same considerations as those made for the regular octahedron for Regions 1 and 4.

The resulting diagram is entirely similar to the one obtained for the octahedron, where the points representing Archimedean polyhedra are aligned in groups of three forming a triangle.

Similar results would be obtained if we begun with the regular dodecahedron owing to the duality relation between this polyhedral shape and the regular icosahedron.

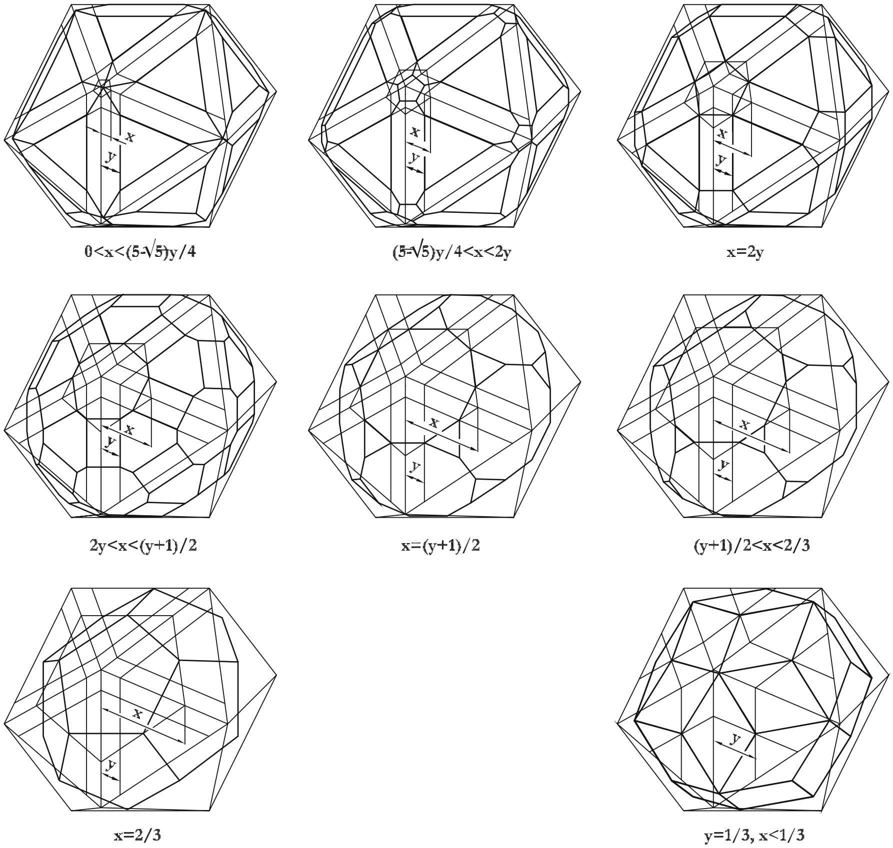

Fig. 6 Possible cases obtained by truncating the vertex and chamfering the edges of the regular icosahedron

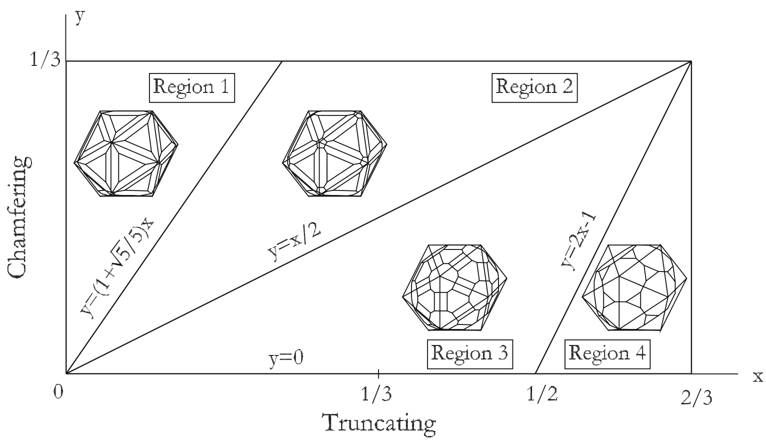

Fig. 7 Truncating and chamfering diagram of the regular icosahedron

Table 2 Truncating and chamfering values to obtain Archimedean polyhedra with icosahedral symmetry

Polyhedron	Truncating (x)	Chamfering (y)
Icosahedron	0	0
Truncated icosahedron	1/3	0
Icosidodecahedron	1/2	0
Truncated icosidodecahedron	$(5-\sqrt{5})/6$	$(3-\sqrt{5})/6$
Rhombicosidodecahedron	$(7-\sqrt{5})/11$	$(7-\sqrt{5})/22$
Truncated dodecahedron	$(15-\sqrt{5})/22$	0
Dodecahedron	2/3	0

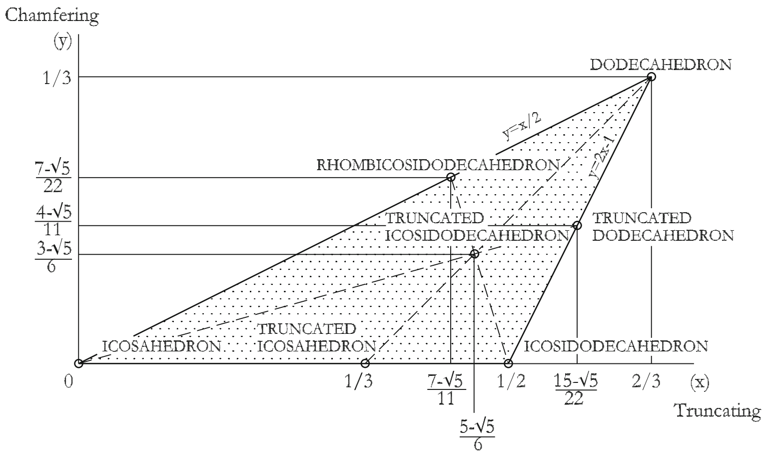

Fig. 8 Situation of the Archimedean polyhedra over the truncating and chamfering diagram

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